MASA student creative thinking quest 2017

The pages which follow show what a teacher might do whilst modelling project behaviour in the classroom and meeting the requirements of the Mathematics syllabus.

Mathematics is a set of processes for constructing, testing and manipulating representations of the quantitative or spatial characteristics of what we perceive.

Performing a mathematical task involves a not necessarily linear combination of experiencing, investigating, conjecturing, believing and proving aspects of the situation from which the task has emerged. (After Jon Roberts 1997)

A problem is a task for which there is no immediately obvious solution, method for finding the solution or way to investigate the situation presented.

In teaching and learning, a project involves combining Mathematics with one or more particular applications to bring together an abstraction and a situation from which it arises or to show how a particular mathematical model arises from several contexts, for example, projectile motion and the inverse square law.

In assessment, a project requires the student to apply, discover or rediscover some Mathematics in order to use it in investigating or solving a problem.

In a competition, a project requires the student, alone or as a member of a group of specified size, to undertake an investigation, possibly using concrete materials, to formulate a problem in mathematical terms, solve it and relate the solution to the original context.

Projects often involve the student work away from the teacher's presence. Having the student keep a journal, answer questions about the task and methods used and observing the development of the project can assist the teacher with having confidence that the product presented is the student's own.

What is the volume of a unit regular tetrahedron?

The origins of the cut-outs referred to below are in David Stonerod's book *Puzzles in Space*,
ISBN 0-914534-03-3

Part 1 serves to introduce the problem and Part 2 explores it further.

1. Fitting a tetrahedron and four pyramids into a cube.

Use the templates to make an open-topped cube, a tetrahedron and four corner pieces.

The four corner pieces and the tetrahedron can be fitted exactly into the cube.

Therefore, the volume of the tetrahedron is the volume of the cube less the volume of the four corner pieces.

2. Determining the volume of a corner piece.

Method 1:

Use the templates to make three of the cube in three parts pieces and six of the cube in six parts pieces.

A cube in three parts piece has the same base area and height as the cube but only one third of its volume.

A cube in six parts piece has the same base area as the cube but only one half of its height and its volume is one sixth of that of the cube.

If base area and height are the only factors to be considered then it is logical to assert that the volume of a pyramid is given by: $Volume = \frac{1}{3} \times area\ of\ base\ \times height$

At some stage it might be productive to generalize this to the general result for pyramids namely, given a pyramid with base area *A* and height *h*, its volume is $\frac{1}{3}AH$ which is one third that of the prism with the same base area and same height.

Method 2 is addressed in Part 4.

3. The sum of consecutive integers and consecutive squares.

In this part, some mathematical steps are taken towards a more rigorous solution to the problem.

The first aim is to find a formula for the sum $1 + 2 + \; 3 + 4 + \cdots + n$

Complete the following table.

n	Sum 1 to n			
1	1	1		
2	1+2	3		
3	1+2+3	6		
4	1+2+3+4	10		
5				
6				
7				
n				

Columns four and five are for you to try something which will produce an easily recognized and useful pattern which can be related to n.

Dividing the sum by *n* shows results on each row one half larger than on the previous row.

On row *n* the number is $\frac{n+1}{2}$ which yields :

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

The second aim is to find a formula for the sum $1^2 + \; 2^2 + \; 3^2 + \cdots + n^2$

n	Sum $1^2 \; to \; n^2$			
1	1^2	1		
2	$1^2 + \; 2^2$	5		
3	$1^2 + 2^2 + \; 3^2$	14		
4	$1^2 + 2^2 \; + \; 3^3 + \; 4^2$	30		
5				
6				
7				
n				

Columns four and five are for you to try something which will produce an easily recognized and useful pattern which can be related to n.

Dividing by the sum of the squares by the sum of the integers shows results on each row which are two thirds larger than on the previous row.

On row *n* the number is $\frac{2n+1}{3}$ which yields:

$$\sum = \frac{n(n+1)(2n+1)}{6}$$

4. Determining the volume of a corner piece.

Method 2:

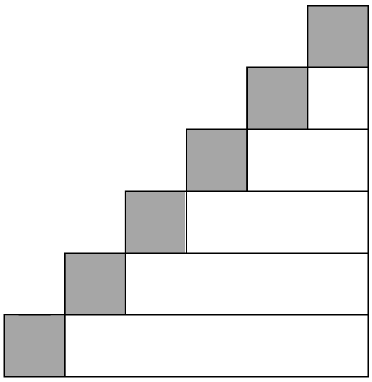
Imagine that a rather lumpy approximation to a corner pyramid is constructed from *n* slices of equal thickness, each parallel to the base.

The profile below represents a cross-section of such a pyramid made from six slices.

The thickness of each slice is: $\frac{1}{6} \times length\ of\ leg\ of\ isosceles\ triangle\ base$
where the leg is one of the perpendicular edges.

Each slice has an isosceles triangular horizontal cross-section with area $\frac{1}{2} \times\ length\ of\ leg^2$

Representing this length by l gives the details which follow



$Volume = \frac{1}{2} \times \left(\frac{l}{6}\right)^2 \times \left(\frac{l}{6}\right)$
$Volume = \frac{1}{2} \times \left(\frac{2l}{6}\right)^2 \times \left(\frac{l}{6}\right)$
$Volume = \frac{1}{2} \times \left(\frac{3l}{6}\right)^2 \times \left(\frac{l}{6}\right)$
$Volume = \frac{1}{2} \times \left(\frac{4l}{6}\right)^2 \times \left(\frac{l}{6}\right)$
$Volume = \frac{1}{2} \times \left(\frac{5l}{6}\right)^2 \times \left(\frac{l}{6}\right)$
$Volume = \frac{1}{2} \times \left(\frac{6l}{6}\right)^2 \times \left(\frac{l}{6}\right)$

If only the light part is considered this gives:

$$V_1 = \frac{1}{2}\left(\frac{l}{6}\right)^2 \frac{l}{6} + \frac{1}{2}\left(\frac{2l}{6}\right)^2 \frac{l}{6} + \frac{1}{2}\left(\frac{3l}{6}\right)^2 \frac{l}{6} + \frac{1}{2}\left(\frac{4l}{6}\right)^2 \frac{l}{6} + \frac{1}{2}\left(\frac{5l}{6}\right)^2 \frac{l}{6}$$

, which is less than the volume of the corner pyramid.

If the dark parts are included, this gives:

$$V_2 = \frac{1}{2}\left(\frac{l}{6}\right)^2 \frac{l}{6} + \frac{1}{2}\left(\frac{2l}{6}\right)^2 \frac{l}{6} + \frac{1}{2}\left(\frac{3l}{6}\right)^2 \frac{l}{6} + \frac{1}{2}\left(\frac{4l}{6}\right)^2 \frac{l}{6} + \frac{1}{2}\left(\frac{5l}{6}\right)^2 \frac{l}{6} + \frac{1}{2}\left(\frac{6l}{6}\right)^2 \frac{l}{6}$$

, which is more than the volume of the corner pyramid.

The volume of the corner pyramid lies between V_1 and V_2

In symbols this says that $V_1 < Volume\ of\ corner\ pyramid < V_2$

As the number of slices used to construct the corner pyramid increases, they become thinner and the piece becomes less lumpy: if the slices are thin enough the corner piece constructed this way can be made as smooth as we please.

When the number of slices is *n* the results are:

$$V_1 = \frac{1}{2}\left(\frac{l}{n}\right)^2\frac{l}{n} + \frac{1}{2}\left(\frac{2l}{n}\right)^2\frac{l}{n} + \frac{1}{2}\left(\frac{3l}{n}\right)^2\frac{l}{n} + \frac{1}{2}\left(\frac{4l}{n}\right)^2\frac{l}{n} + \cdots + \frac{1}{2}\left(\frac{(n-1)l}{n}\right)^2\frac{l}{n}$$

$$= \frac{1}{2}\left(\frac{l}{n}\right)^3 (1^2 + 2^2 + 3^2 + \cdots + (n-2)^2 + (n-1)^2)$$

$$= \left(\frac{1}{2}\right)\left(\frac{l}{n}\right)^3 \frac{(n-1)n(2n-1)}{6}$$

$$= \frac{l^3}{12}\left(1-\frac{1}{n}\right)\left(\frac{n}{n}\right)\left(2-\frac{1}{n}\right)$$

Notice that, as n is made larger and larger the right hand side becomes closer and closer to $\frac{l^3}{6}$ but it is smaller.

For V_2 we have:

$$V_2 = \frac{1}{2}\left(\frac{l}{n}\right)^2\frac{l}{n} + \frac{1}{2}\left(\frac{2l}{n}\right)^2\frac{l}{n} + \frac{1}{2}\left(\frac{3l}{n}\right)^2\frac{l}{n} + \frac{1}{2}\left(\frac{4l}{n}\right)^2\frac{l}{n} + \cdots + \frac{1}{2}\left(\frac{(n-1)l}{n}\right)^2\frac{l}{n} + \frac{1}{2}\left(\frac{nl}{n}\right)^2\frac{l}{n}$$

$$= \frac{1}{2}\left(\frac{l}{n}\right)^3 (1^2 + 2^2 + 3^2 + \cdots + (n-2)^2 + (n-1)^2 + n^2)$$

$$= \left(\frac{1}{2}\right)\left(\frac{l}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{l^3}{12}\left(\frac{n}{n}\right)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

Notice again that, as n is made larger and larger, the value of the right hand side becomes closer and closer to $\frac{l^3}{6}$ but it is larger.

Since the values of both V_1 and V_2 become closer and closer to $\frac{l^3}{6}$ as n is made larger and larger and the volume of the corner pyramid lies between V_1 and V_2 it is logical to conclude that the volume of the corner pyramid is $\frac{l^3}{6}$

5. The volume of a unit tetrahedron.

The corner pieces have half the base area and the same height as the cube so their volume appears to be one sixth that of the cube and so the volume of the tetrahedron is one third that of the cube.

Given that the tetrahedron has edge length $\sqrt{2}\ that\ of\ the\ cube$ its volume will be $2\sqrt{2}\ that\ of\ the\ cube$

Hence the volume of a unit cube is: $\frac{1}{2\sqrt{2}} \times \frac{1}{3} = \frac{\sqrt{2}}{12}$

Further activities based on these approaches include:

- What is the volume of an octahedron?
- What is the volume occupied by two intersecting tetrahedra?
- What is the volume of a stella-octangula?
- The properties of the cut-outs used in these activities.
- Inductive proofs of sums formulae.
- Areas and volumes by counting squares or cubes and determining limits.
- Deduce the formulae for the surface area and volume of a sphere.
- Deduce the volume of an ellipsoid or paraboloid.
- Determine the sums of consecutive odd numbers and multiples.
- Determine the sums of consecutive cubes and fourth powers.
- Can a shape be folded to make the surface of a solid?

Cut-outs

- If the cut-outs are photocopied they can be rearranged to fit more than one to a page.
- Initially, make just one copy from your master and check that the length of the edges of the squares is 5cm. If it is not, adjust the photocopier's enlargement/reduction setting.
- Using adhesive tape along the edges may help in finishing the solid.
- If you copy onto thin card to make more robust solids, score along the folding edges with a compass point and ruler.
- Copying onto coloured material can assist in distinguishing the different solids and even add an aesthetic dimension to the students' work.
- To make a unit tetrahedron, photo-reduce the tetrahedron template from A3 to A4.

Templates for cut-outs

Open topped cube

Cut out the shape below.

Cut along the heavy lines.

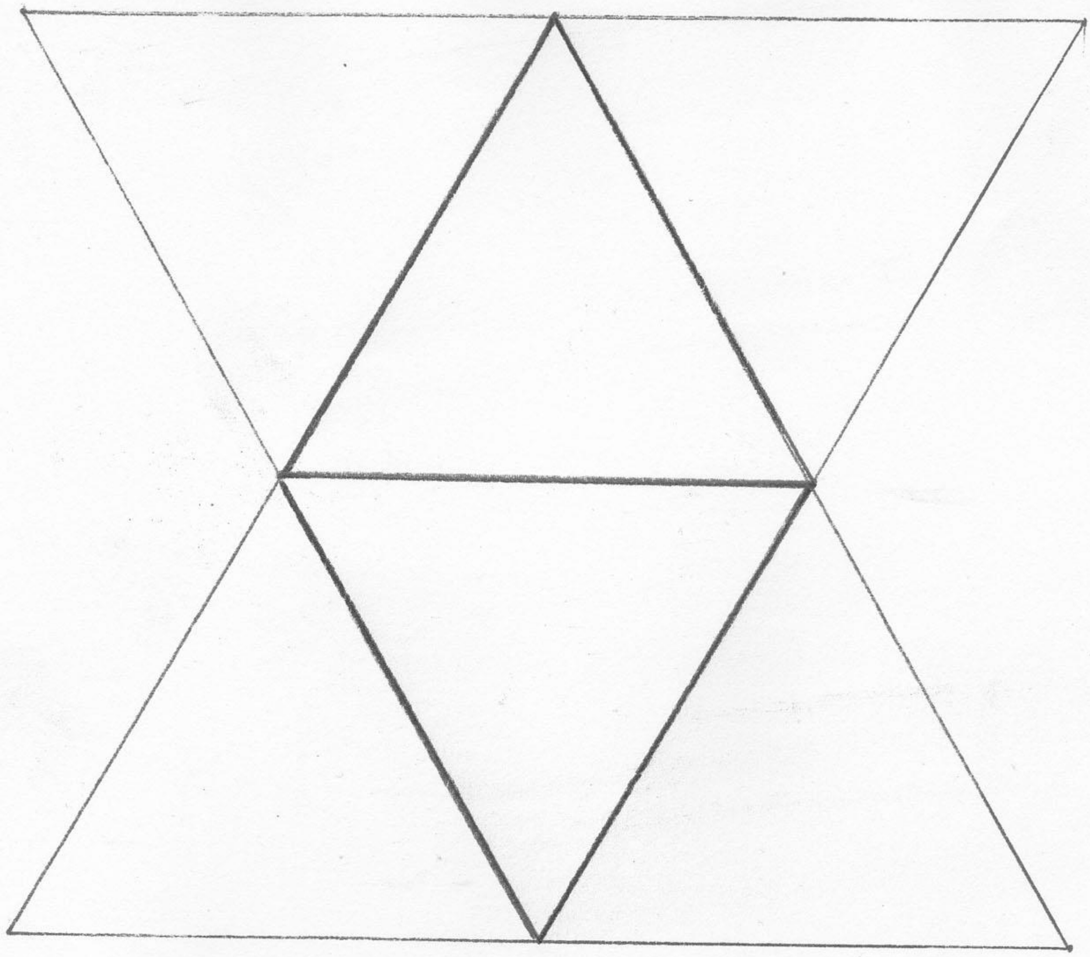
Fold and glue the flaps.

Tetrahedron

Cut out the shape below.

Fold along the heavy lines.

Glue the flaps.



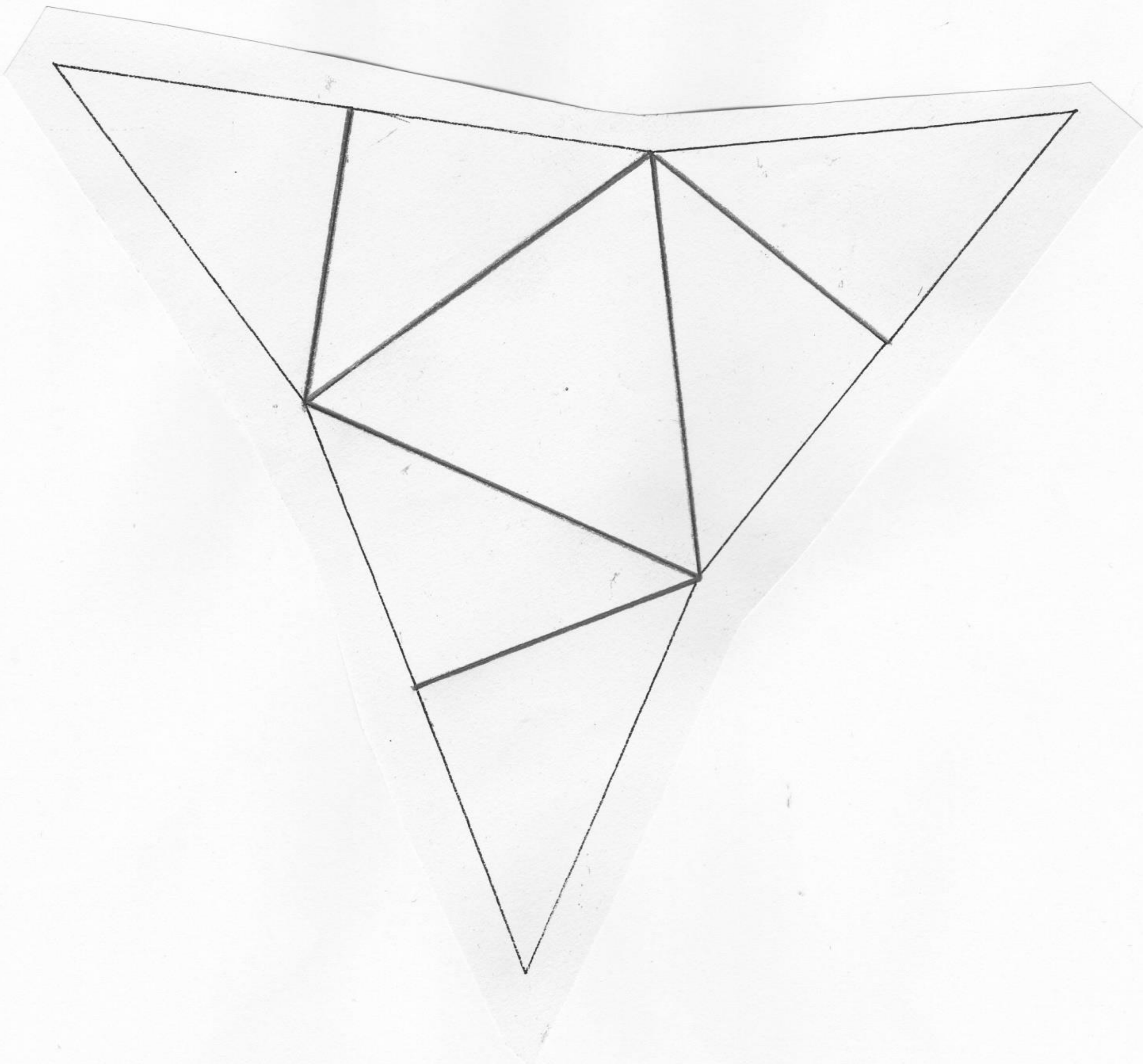
Corner pyramid

You need four of these.

Cut out the shape below.

Fold along the heavy lines.

Glue the flaps.



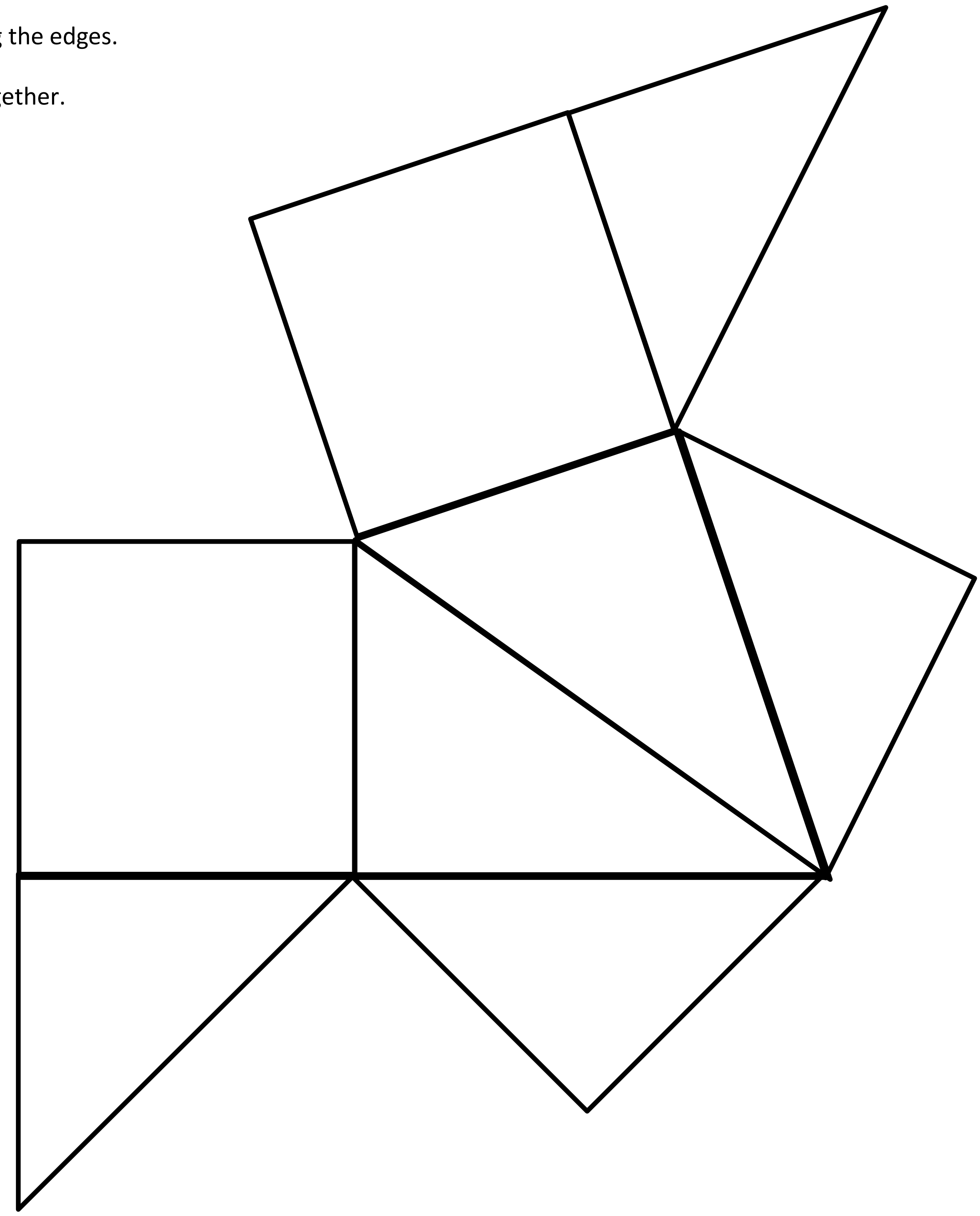
Cube in three parts

You will need three of these.

Cout out the shape.

Fold sharply along the edges.

Glue the faces together.



 ©MGW2016

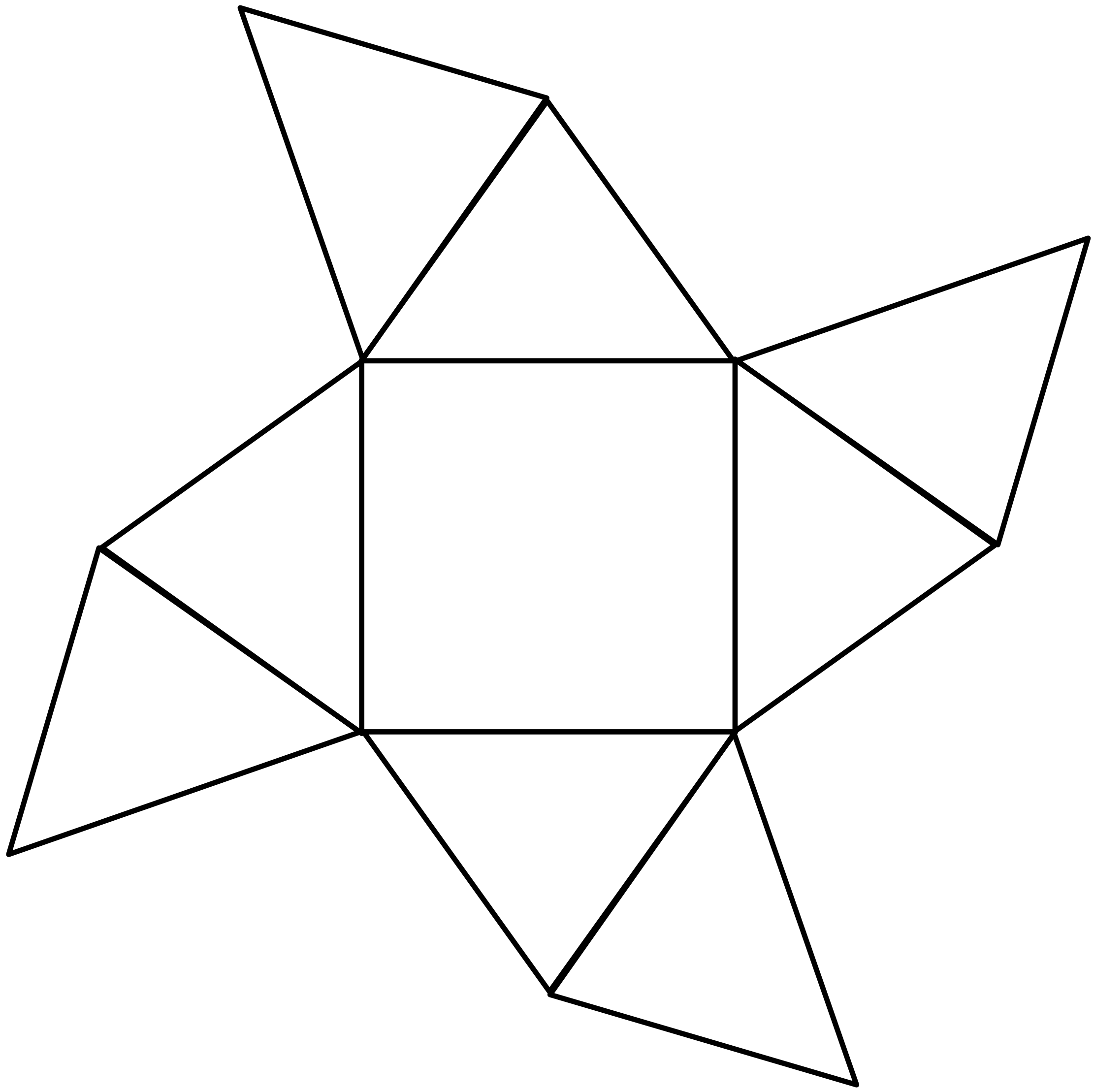
Cube in six parts

You will need six of these.

Cut out the shape.

Fold along each of the lines.

Glue the flaps.



 ©MGW2016