

# HOW LONG WILL THE WATER LAST?

## The modelling process in the classroom

The following example, adapted from the AAMT 1994 Mathematics Week publication by Henry & McAuliffe (1994), entitled *Drying Out* and uses a structured modelling approach.

### Stage 1: Real-world problem

The problem: *There are many lakes in Australia that are dry for most of the time, only filling for short periods immediately after rain. Lake Eyre in South Australia is an example of one of these normally dry lakes. When a dry lake bed is filled with water, how rapidly will the lake empty?*

The problem statement should be very general and free of as much data as possible, as later stages of the modelling process will consider and gather what is needed. Often teachers will have to rewrite a typical textbook problem to reduce the initial information included.

Suggested classroom techniques include:

- Allowing students individually to read and think about the problem statement.
- Dividing the class into groups and allowing them to discuss the problem. Some classes may need the structure of focusing questions for their discussion. A good technique is to get the students to list key words, to restate the problem, and to determine what units will be used to express the answer.
- Listing key words: *shallow, empty, filled.*
- Restating problem: *Find how long it takes the lake to become dry again. We expect the answer to be in days.*

### Stage 2: Make assumptions

This may be regarded as the most valuable part of the process and it should not be rushed. It consists of listing all the variables involved and then trying to simplify or modify the list. In this process, it becomes obvious that there is a need to obtain certain information that will constitute the initial conditions of the problem.

Suggested classroom techniques include:

- Individually, or in groups, have the students brainstorm the problem to uncover the variables involved. The teacher could collect and write the variables on the board.
- The teacher could then lead a class discussion in which each variable is considered and is either regarded as being important to the model or is discarded or controlled. This discarding process is really a way of controlling the model for certain variables and thus reducing the complexity of the model. For example, if the problem involves cars, then we can reduce the variables to be considered, such as size, make and colour, by limiting our model to a particular car. Average rates can also help to overcome a host of seasonal variables but can cause your model to be less sensitive.
- During this stage, the class also compiles a list of the initial information that is required. The students could be directed to do some research to find this information. Commonwealth and State Government publications are useful sources. The use of simulations, databases or spreadsheets can also provide the raw data from which the students can select the data they regard as important.

The following is a list of variables and the results of their consideration:

- size, shape, depth, surface area of water: *will depend upon initial conditions, thus more information is needed.*
- temperature, evaporation: *will depend partly upon surface area; more information is needed.*
- seepage: *will depend upon surface area of bed and type of bed surface; more information is needed.*

- rainfall, seepage, ground water, etc.: *It was decided that there will be no more input for this lake for the period of our study.*
- vegetation, irrigation: *Because of desert conditions these are unlikely to be significant.*
- animals: *because of desert conditions, unlikely to be significant*
- salinity: *controlled by using average evaporation rates*
- runoff: *Lake Eyre is the lowest point, therefore nil runoff.*

### Stage 3: Formulate mathematical problem

The choice of mathematical model will depend upon the approach used by the teacher and the demands of the subject. If the class decides upon a model that does not match the teacher's, then the teacher has a choice to either intervene (a structured approach) or to delay until the completion of one cycle of the modelling process. The teacher has the option of algebraically constructing the model with the class or providing a spreadsheet containing the model.

Suggested classroom techniques include:

- Groups discussing the mathematics they can use to make sense of the problem and reporting back to the class. For example, if the situation or problem includes a collection of data, then the use of various graphs may be an adequate model.
- Listing the calculations that need to be made.
- Setting up the spreadsheets so that the first line contains the headings, the second contains the numerical values of the initial conditions and calculations completed, and the third contains the equations used for the calculations. The *fill down* command will then extend the spreadsheet as far as you wish to go. This technique allows the initial conditions to be altered without redoing the spreadsheet and permits many investigations by the variation of the initial values.

*Initial data (provided by Henry & McAuliffe):*

*Shallow lake dimensions: the ratio of maximum depth to maximum radius = 0.0001  
Volume of Lake Eyre = 1 000 000 cu m  
Evaporation rate per day = 0.003 cu m/ sq m of surface area  
Seepage rate per day = 0.0008 cu m/ sq m of lake bed.*

*Now some may feel uneasy with using the volume as 1 000 000. Why not use a realistic figure, some may ask? Well at some stage the volume of the lake will be 1 000 000, but it is easier to begin with this figure as later, when the spreadsheet is set up, you can use any input and it will automatically adjust.*

*To select the model of lake: The best model is probably a very flat cylinder; however, we will use a flat cone as proposed by Henry & McAuliffe because it raises some interesting discussion in stage 6 of the process.*

*With some classes, you may decide to round off certain data (which is the case for the solution that follows). If this is done, when you reach stage 6 in the modelling process, you can go back and undo this and see how much the result is changed.*

*Discussion: The class will determine that the calculations needed are:*

- *surface area of lake*
- *surface area of lake bed*
- *radius of the lake*
- *loss of volume/depth due to evaporation*
- *depth of the lake*
- *loss of volume/depth due to seepage*

<b>Lake model</b>							
Depth ratio	0.0001		Value of Pi	3.141592654			
Evaporation rate	0.003	cu m/sq m					
Seepage rate	0.0008	cu m/sq m					
Day Number	Volume	Radius	Depth	Surface area	Lake bed area	Volume loss	
	0	1000000	2121.569	0.212156884	14140479.2	14140479.27	53733.821
	1						
	2						
	3						

#### Stage 4: Solve the mathematical problem

This stage describes the process used by the students when applying a procedure to given data. Using the modelling process may mean a return to the initial assumptions in order to modify the problem being considered.

*Using the fill down option on the spreadsheet, we can continue until we get negative values for the volume of the lake. Note that we are really using an iterative "salami slicing" technique, whereby the spreadsheet calculates the volume loss at the end of the day, then adjusts and repeats the calculations for the new volume. The spreadsheet formulae can be found in Appendix 1.*

<p>Volume</p> $V = \pi r^2 d$ $V = \frac{1}{3} \pi r^2 (0.0001r)$ $V = \frac{0.0001}{3} \pi r^3$ $\frac{3 \times V}{0.0001\pi} = r^3$ $r = \sqrt[3]{\frac{3V}{0.0001\pi}}$	<p>Depth</p> $d = 0.0001r$	<p>Surface area (SA) of lake</p> $= \pi r^2$	<p>Surface area (SA) of lake bed = <math>\pi s</math> where</p> $s = \sqrt{r^2 + (0.0001r)^2}$	<p>Volume loss =</p> <p>Evaporation + Seepage</p> $= 0.003SA \text{ of lake} + 0.0008SA \text{ of lake bed}$
--	----------------------------	--	--	--

Lake model						
Depth ratio	0.0001		Value of Pi	3.141592654		
Evaporation rate	0.003	cu m/sq m				
Seepage rate	0.0008	cu m/sq m				
Day Number	Volume	Radius	Depth	Surface area	Lake bed area	Volume loss
0	1000000	2121.569	0.212156884	14140479.2	14140479.27	53733.821
1	946266.2	2082.867	0.208286713	13629282.9	13629282.96	51791.2751
2	894474.9	2044.152	0.204415197	13127324.9	13127324.97	49883.8347
3	844591.1	2005.423	0.200542283	12634608.35	12634608.41	48011.5118
4	796579.6	1966.679	0.196667915	12151136.48	12151136.54	46174.3187
5	750405.2	1927.92	0.192792032	11676912.67	11676912.73	44372.2682
6	706033	1889.146	0.18891457	11211940.45	11211940.5	42605.3737
7	663427.6	1850.355	0.185035463	10756223.46	10756223.51	40873.6492
8	622553.9	1811.546	0.181154638	10309765.53	10309765.58	39177.109
9	583376.8	1772.72	0.177272018	9872570.624	9872570.673	37515.7684
10	545861.1	1733.875	0.17338752	9444642.907	9444642.954	35889.6431
11	509971.4	1695.011	0.169501056	9025986.717	9025986.762	34298.7496
12	475672.7	1656.125	0.165612532	8616606.596	8616606.639	32743.1051
13	442929.6	1617.218	0.161721845	8216507.302	8216507.343	31222.7278
14	411706.8	1578.289	0.157828885	7825693.822	7825693.861	29737.6366
15	381969.2	1539.335	0.153933536	7444171.392	7444171.429	28287.8513
16	353681.4	1500.357	0.150035668	7071945.515	7071945.55	26873.393
17	326808	1461.351	0.146135144	6709021.982	6709022.015	25494.2836
18	301313.7	1422.318	0.142231812	6355406.894	6355406.926	24150.5462
19	277163.1	1383.255	0.138325511	6011106.692	6011106.722	22842.2055
20	254320.9	1344.161	0.13441606	5676128.183	5676128.211	21569.2871
21	232751.6	1305.033	0.130503265	5350478.572	5350478.599	20331.8186
22	212419.8	1265.869	0.126586913	5034165.5	5034165.525	19129.8289
23	193290	1226.668	0.122666767	4727197.084	4727197.108	17963.3489
24	175326.6	1187.426	0.118742567	4429581.966	4429581.988	16832.4115
25	158494.2	1148.14	0.114814027	4141329.361	4141329.382	15737.0516
26	142757.2	1108.808	0.110880825	3862449.121	3862449.14	14677.3067
27	128079.9	1069.426	0.106942605	3592951.802	3592951.82	13653.2169
28	114426.7	1029.99	0.102998966	3332848.744	3332848.761	12664.8252
29	101761.8	990.4946	0.099049458	3082152.161	3082152.176	11712.1782
30	90049.65	950.9357	0.095093568	2840875.247	2840875.262	10795.326
31	79254.33	911.3072	0.091130718	2609032.304	2609032.317	9914.32276
32	69340.01	871.6024	0.08716024	2386638.882	2386638.894	9069.22776
33	60270.78	831.8137	0.083181368	2173711.959	2173711.97	8260.10545
34	52010.67	791.9321	0.079193211	1970270.144	1970270.154	7487.02656
35	44523.65	751.9472	0.075194724	1776333.929	1776333.938	6750.06894
36	37773.58	711.8467	0.071184674	1591925.99	1591925.998	6049.31877
37	31724.26	671.6159	0.067161588	1417071.564	1417071.571	5384.87195
38	26339.39	631.2369	0.063123686	1251798.906	1251798.912	4756.83585
39	21582.55	590.6879	0.059068787	1096139.877	1096139.883	4165.33154
40	17417.22	549.9418	0.054994181	950130.6824	950130.6872	3610.4966
41	13806.72	508.9643	0.050896431	813812.8289	813812.8329	3092.48875
42	10714.23	467.7109	0.046771091	687234.3853	687234.3887	2611.49067
43	8102.743	426.1225	0.042612252	570451.6645	570451.6673	2167.71633
44	5935.027	384.118	0.038411803	463531.5124	463531.5147	1761.41975
45	4173.607	341.5815	0.034158149	366554.469	366554.4709	1392.90698
46	2780.7	298.338	0.029833797	279619.157	279619.1584	1062.5528
47	1718.148	254.1035	0.025410348	202848.1726	202848.1736	770.823057
48	947.3245	208.3643	0.020836433	136394.4314	136394.4321	518.29884
49	429.0257	160.0116	0.016001163	80436.46416	80436.46456	305.658564
50	123.3671	105.6145	0.010561451	35042.65615	35042.65633	133.162094
51	-9.795	-45.3933	-0.00453933	6473.419318	-6473.419351	14.2415225

## Stage 5: Interpret the solution

After obtaining their solutions, the students are directed back to the problem. They must check to ensure that they have answered the problem within the assumptions they have made. Interpretations made should make explicit the assumptions and initial conditions. This is a crucial step in helping students realise that solutions to problems are constrained by the context and are not easily transferable to other situations.

*The students will discuss the results and decide whether the answer is the last day water was in the lake or the first dry day. Therefore, our answer is not just 50 days. It requires an answer of the type: If the volume of our lake was ... etc. then it is empty on the 50th day*

## Stage 6: Verify the model

In this stage, the strengths and weaknesses of the model are discussed. This involves reflecting upon the mathematics that has been used. The statement that “all models are wrong, but some are useful” is an important reminder of the dangers of oversimplification and of ignoring the underlying assumptions. Models should be evaluated in terms of the variables used and, more importantly, those omitted.

Suggested classroom techniques include:

- Exploring ways of improving the model by considering more variables and carrying out further simulations. By varying initial conditions in the spreadsheet, the class can examine their effect.
- Repeating the process with a new refined model. In considering the process of improving the model, the more variables that can be included, the more sensitive the model will be to the context of the original problem.

*The class can consider different models from the flat cone, and consider different variable values. This is extremely easy to do because it merely involves minor adjustments to the spreadsheet. The students can experience the power of their mathematics in making predictions and improving their model. The students also discuss their mathematical understanding, and this provides the teacher with an opportunity to introduce more sophisticated mathematics.*

## Stage 7: Report, explain, predict

This is a valuable part of the process, as students need experience in using language to express mathematical ideas. It is here that we reflect upon the quality of the students’ thinking. It should include documentation of the students’ progress through the stages of the cycle as well as their final predictions and answers.

The structure of the modelling process provides a good organising device for their report.

Suggested classroom techniques include:

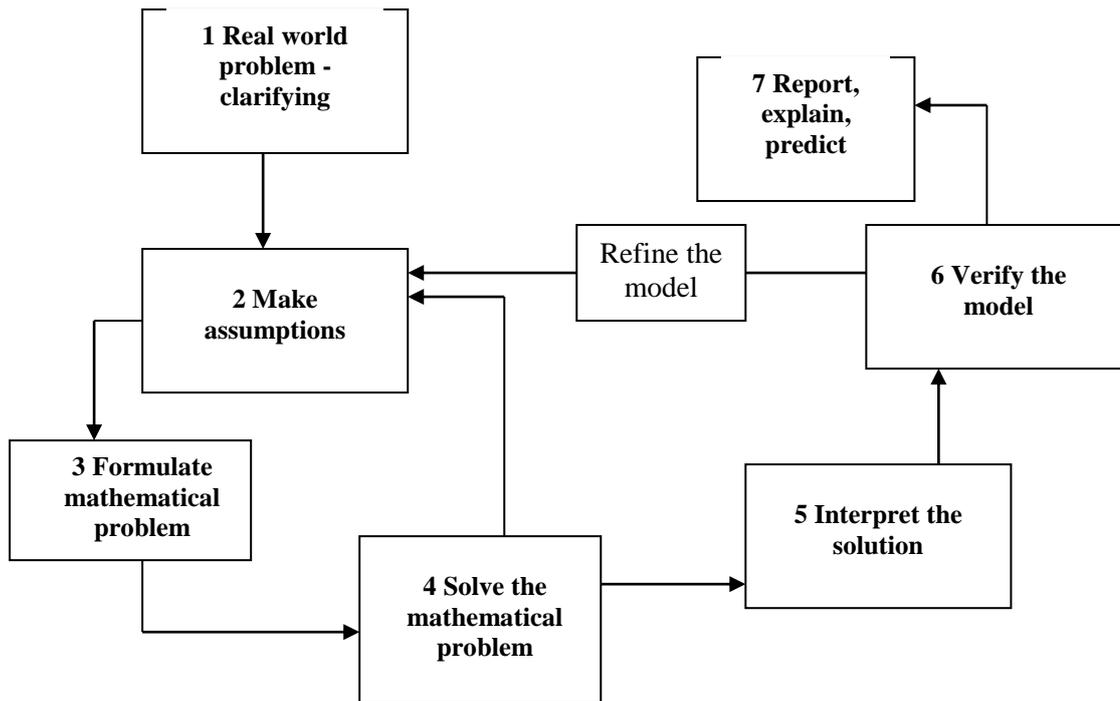
- Keeping a problem field book which keeps a running record of students’ thinking. Problems, possibilities and solutions are all recorded. This is a valuable source for later reports.
- A report prepared by the class, either individually or in groups. This can take many forms and can be used for display and assessment purposes.
- A class discussion of the reports. This is a valuable opportunity for students to think and to discuss their mathematics as well as propose further investigations.
- Further manipulation of the model, where various theories can be raised and investigated.
- The use of simulations, databases or spreadsheets can be very useful at this stage. Students could be encouraged to vary the initial conditions and their assumptions concerning the variables to observe the effect upon the answers.

## Conclusion

Obviously, mathematical modelling problems don’t always require a computer or a spreadsheet. However, scattered throughout the Mathematics Stage 1 and 2 Subject Outlines are suggestions on

using spreadsheets. It has long been acknowledged that students can learn mathematical modelling only by participating in the experience of constructing models and in the struggle to perfect their models.

This requires the teacher to allow the students to formulate, to test, to discuss and to adjust their thinking. The role of the teacher is to provide the opportunity for this learning to take place and to guide the students through the modelling process, while allowing them freedom within each stage. No amount of looking at, or of applying, models made by others can produce the confidence and skills that flow from making one's own models, no matter how crude. The open modelling approach, rather than an application approach to modelling, is needed if we are to equip our students with the confidence needed to apply mathematics in their day-to-day lives.



*The modelling process*

## Appendix

### Spreadsheet formulae

- Radius of the lake: Cell C8= $(3*B8/(3.141592654*B$3))^{1/3}$
- Depth of the lake: Cell D8 = $C8*B$3$
- Lake surface area: Cell E8 = $E$3*C8^2$
- Lake bed surface area: Cell F8 = $E$3*C8*(C8^2+(B$3*C8)^2)^{1/2}$
- Volume/depth loss due to evaporation and seepage: Cell G8 = $E8*B$4+F8*B$5$

<http://www.k26.com/eyre/index.html>





